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Lewis Research Center
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SUMMARY

Explicit solutions are obtained in parametric form for the magnetic field due to surface currents flowing on an infinitely long rectangular conductor. The well-known analogy between the two-dimensional magnetic potential problem and the corresponding electrostatic problem is exploited to obtain these solutions. The method employed is a previously developed Schwartz-Christoffel transformation that maps the complex space outside a rectangle into the complex space outside a unit circle. Explicit solutions for the magnetic field outside the rectangle are derived.

INTRODUCTION

The development of small electronic systems has stimulated considerable interest in the use of conductors with rectangular cross sections, such as thin films, tapes, etc. Application of these configurations requires a knowledge of the magnetic field created by the currents passing through the systems. In ordinary conductors where the current is uniformly distributed, the magnetic field can be obtained by straightforward application of the Biot-Savart law, but if the current only flows on the surface of the samples, such as with superconductors or with rapidly oscillating fields, minimum energy configurations must be considered, and the problem becomes more complex. In recent years, attempts to use a Green's function technique to solve for the current density in superconductors (refs. 1 and 2) led to an integral equation that could not be integrated explicitly. Solutions are generally obtained by numerical integration using successive approximations. A simple alternative approach is used herein. This approach is based on the fact that a good approximation of the magnetic field configuration around the rectangle can be found at distances larger than the penetration or skin depth away from the conductor by assuming that all currents flow on the surface of the conductor. With this assumption, a qualitative description of the current distribution can be obtained. This assumption is valid particularly for superconductors and normal conductors at high frequencies.

For the case considered herein, infinitely long conductors of constant rectangular cross section with the current flowing only on the surface (e.g., a perfect conductor), a close connection between some electrostatic and magnetic problems exists. Knowledge of the relation between the electric and magnetic field potential functions in these instances allows the use of techniques developed for solutions of Laplace's equation, such as complex transformations. Through an analogy between the corresponding potential functions, explicit equations are derived for the magnetic field and surface current distribution by use of the conformal transformation of a rectangle into a unit circle (ref. 3). The resulting field is described by equations in which the parameters are the coordinates of an intermediate space.

ANALYSIS

Fundamental Mapping

For a conductor of zero resistivity, no internal electric field can be maintained. Maxwell's equation for a time-varying magnetic field $\nabla \times \vec{E} = -(\partial\vec{B}/\partial t)$ then leads to the conclusion that the magnetic field $\vec{B}$ is a function of coordinates alone. In order to simulate the case of superconductors, the Meissner condition that $B = 0$ inside is used. This case also applies to normal conductors with high-frequency alternating currents. Since no magnetic or electric field is allowed to exist in the interior of this perfect conductor, the current density is also zero, as indicated by Maxwell's equation for a time-varying electric field $\nabla \times \vec{B} = \vec{J} + \epsilon(\partial\vec{E}/\partial t)$. At the boundary, however, surface currents and surface charges are possible. The tangential component of magnetic field is discontinuous at the surface of the conductor by an amount proportional to the surface current per unit width. The normal component of magnetic field is continuous across the surface, and since it is zero inside the perfect conductor, it must go to zero outside the boundary. In analogy, the normal component of electric field is discontinuous at the surface by an amount proportional to the surface-charge density. If the perfect conductor is assumed to be sufficiently long so that the distribution of current is not affected by contacts, a two-dimensional analysis is possible.

As shown in some texts on electromagnetic theory (ref. 4), an analogy exists between the magnetic potential $\vec{A}$ due to a surface current and the electrostatic potential φ due to a surface-charge density. For a conductor whose axis is taken in the direction of the z-axis, the basic solutions for each type of problem are as follows:

Electrostatic:

$$\varphi = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(x', y')dv'}{|\vec{r} - \vec{r}'|}$$

$$\tag{1}$$

Magnetic:

$$\vec{A} = \hat{k}A_z = \hat{k}\,\frac{\mu_0}{4\pi} \int \frac{J_z(x', y')dv'}{|\vec{r} - \vec{r}'|}$$

(All symbols are defined in the appendix.) Thus, if ρ_z and J_z are similar functions of the coordinates, then φ and A_z (as well as $\vec{E}$ and $\vec{B}$) will have similar solutions provided that φ and A_z satisfy similar boundary conditions.

The necessary and sufficient conditions to define a unique potential φ in all space are

$$\varphi_1 = \varphi_2 + \text{constant}$$

and

$$\kappa_2 \frac{\partial\varphi_2}{\partial n} - \kappa_1 \frac{\partial\varphi_1}{\partial n} = -\frac{\sigma}{\epsilon_0}$$

where φ_1 and φ_2 are the potential functions in regions 1 and 2, respectively, κ_1 and κ_2 are the dielectric constants, and σ is the surface-charge density.

The equivalent boundary conditions for A_z are derived as follows: If $\vec{B}_1$ and $\vec{B}_2$ are the magnetic fields inside and outside the rectangle, respectively, and $\hat{n}$ is a unit vector perpendicular to the surface, the following equation results from $\vec{\nabla} \cdot \vec{B} = 0$:

$$(\vec{B}_2 - \vec{B}_1) \cdot \hat{n} = 0$$

where

$$\vec{B} = \overrightarrow{\nabla A_z} \times \hat{k}$$

Then, since $\vec{B} = \overrightarrow{\nabla A_z} \times \hat{k}$,

$$(\overrightarrow{\nabla A}_{z,2} \times \hat{k} - \overrightarrow{\nabla A}_{z,1} \times \hat{k}) \cdot \hat{n} = 0$$

which for the rectangular coordinate system leads to

$$\hat{n} \times (\overrightarrow{\nabla A}_{z,2} - \overrightarrow{\nabla A}_{z,1}) = 0 \quad ,$$

and

$$A_{z,2} - A_{z,1} = \text{constant}$$

The secondary boundary condition on A_z is obtained as follows:

$$\hat{n} \times \left(\frac{\vec{B}_2}{\mu_2} - \frac{\vec{B}_1}{\mu_1} \right) = \vec{J}$$

or

$$\hat{n} \times \left(\frac{1}{\mu_2} \overrightarrow{\nabla A}_{z,2} \times \hat{k} - \frac{1}{\mu_1} \overrightarrow{\nabla A}_{z,1} \times \hat{k} \right) = \vec{J}$$

which for the rectangular coordinates leads to

$$\left[\hat{n} \cdot \left(\frac{1}{\mu_2} \overrightarrow{\nabla A}_{z,2} - \frac{1}{\mu_1} \overrightarrow{\nabla A}_{z,1} \right) \right] \hat{k} = - \vec{J}$$

and finally

$$\left(\frac{1}{\mu_2} \frac{\partial A_{z,2}}{\partial n} - \frac{1}{\mu_1} \frac{\partial A_{z,1}}{\partial n} \right) \hat{k} = - \vec{J}$$

Since both A_z and φ obey Laplace's differential equation inside and outside and are subject to the same boundary conditions, the uniqueness theorem indicates that the solutions for the two potential fields are identical (ref. 5) hence, $A_z = C_0 \varphi$ where C_0 is a constant used to obtain the proper units. Now

$$\vec{E} = - \text{grad } \varphi = - \frac{1}{C_0} \text{grad } A_z \tag{2}$$

and

$$\vec{B} = C_0 \hat{k} \times \vec{E} \tag{3}$$

Equation (3) shows that $\vec{B}$ and $\vec{E}$ are perpendicular to each other at any point. Since

$$A_z = \frac{\mu_o}{4\pi} \int \frac{J_z(x', y')}{|\vec{r} - \vec{r}'|} dv' = C_0 \frac{1}{4\pi\epsilon_o} \int \frac{\rho(x', y')}{|\vec{r} - \vec{r}'|} dv' \tag{4}$$

then

$$J_z(x', y') = \frac{C_0}{\mu_o \epsilon_o} \rho(x', y') \tag{5}$$

where $J_z(x', y')$ is a surface current density and $\rho(x', y')$ is a surface-charge density. Since the analysis is two-dimensional, J_z is a current per unit width, and $\rho(x', y')$ is a charge per unit width per unit length. Hence, two-dimensional magnetic problems allowing surface currents only can be solved by using techniques common to two-dimensional electrostatic potential problems. One such method involves the use of a conformal transformation of complex spaces. This transformation facilitates the solution of Laplace's equation by mapping into a space of simpler geometry.

In solving potential problems by use of the conformal transformation, it is often assumed that, once the transformation equations are derived, the solution is essentially complete. Certain parameters of the problem, such as the charge distribution at the surface and the shape of the equipotentials, may be found directly from the transformation equation; however, the dependence of the field and of the potential on distance and angular inclination from the source is still unknown. The procedure to follow then is to solve Laplace's equations for the simpler geometry and then map the solutions into the more complex geometry by using the Schwartz-Christoffel transformation equations. Specifically, the actual field and potential for a desired location are obtained by mapping the proper points outside the simple conductor into the desired point and by solving for the associated field and potential.

Solutions for the Rectangular Conductor

Consider an infinitely long conductor of uniform rectangular cross section with sides 2a and 2b (fig. 1). Current flow through the system is assumed to exist only in an infinitesimally thin layer at the surface. The potential around the rectangle can be found with the use of the Schwartz-Christoffel transformation. A transformation equation was derived in reference 3 that maps the two-dimensional region outside a rectangle in the complex ζ-space (fig. 1) into the two-dimensional region outside a unit circle in the complex t-space. The following differential equation is derived in reference 3 to perform the transformation:

$$\frac{d\zeta}{d\tau} = F'(2 \cos 2\tau - 2\mu)^{1/2} \tag{6}$$

where $t = e^{i\tau}$. The parametric solution of this equation $x = f(\tau)$ and $y = g(\tau)$ described in reference 3 is not in a form that is easily usable; however, since tables and computer programs of Legendre polynomials are now available, a more readily usable solution may be derived in the following manner. Integration of equation (6) leads to

$$\zeta = 2F' \sum P_n(\mu) e^{-i(2n+1)\tau} \left[\frac{-i(2n+1)\cos 2\tau + 2 \sin 2\tau}{4 - (2n+1)^2} - \frac{i\mu}{2n+1} \right] + D \tag{7}$$

where $P_n(\mu)$ are the Legendre polynomials. The constant of integration D corresponds to a displacement of the circle to which the rectangle is mapped from the origin of the t-space, and for simplicity is set equal to zero. The interpretation of $\mu \equiv \cos 2\alpha$ in the t-space is extremely important. Each point on the circumference of the circle given

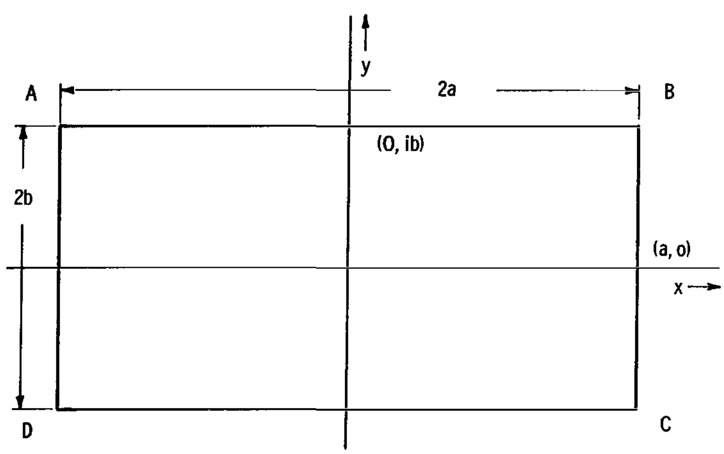


Figure 1. - Rectangular cross section showing ζ plane.

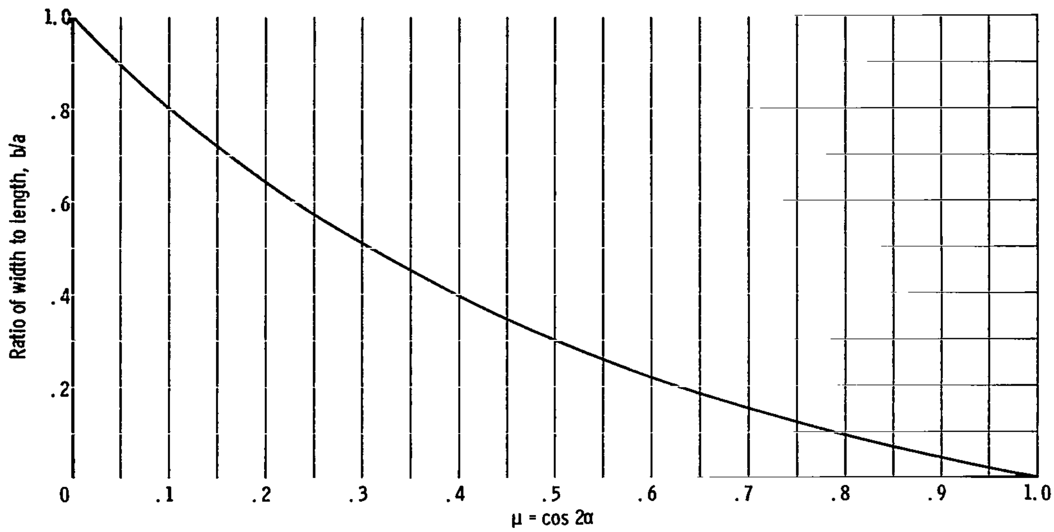


Figure 2. - Ratio of thickness to width against argument of Legendre's polynomials.

by $1\pm\alpha$ and $\pi\pm\alpha$ maps into a corner of the rectangle. Therefore, α is a constant whose value dictates the ratio of the sides of the rectangle, that is, b/a. The relation between μ and b/a is shown in reference 3 to be expressible in parametric form:

$$\frac{b}{a} = \frac{E(k) - (1 - k^2)K(k)}{E\left(\sqrt{1 - k^2}\right) - k^2 K\left(\sqrt{1 - k^2}\right)}$$

and

$$\mu = 1 - 2k^2$$

This relation is plotted in figure 2. Solving equation (7) in terms of the components of τ, with $\tau = \theta - i\beta$, yields the following relation between the two complex planes:

$$\begin{aligned}\zeta = 2F' \sum_{n=0}^{\infty} P_n(\mu)e^{-(2n+1)\beta}\Big[\cos(2n + 1)\theta - i \sin(2n + 1)\theta\Big] \\ \times\left\{\frac{\sin 2\theta}{4 - (2n + 1)^2}\Big[2 \cosh 2\beta + (2n + 1)\sinh 2\beta\Big]\right. \\ \left. - \frac{i \cos 2\theta}{4 - (2n + 1)^2}\Big[2 \sinh 2\theta + (2n + 1)\cosh 2\beta\Big] - \frac{i\mu}{2n + 1}\right\} \qquad (8)\end{aligned}$$

where the complex space ζ can be broken into components: $\zeta = x + iy$. The constant F', as derived in reference 3, is $F' = ib/2L$ where $L \equiv E - k'K$ and K and E are elliptical integrals of the first and second kind, respectively. From equation (8) the parametric equations for x and y can be written as

$$x = \frac{b}{L}\sum_{n=0}^{\infty} P_n(\mu)e^{-(2n+1)\beta}\left\{\cos(2n + 1)\theta \cos 2\theta \left[\frac{2 \sinh 2\beta + (2n + 1)\cosh 2\beta}{4 - (2n + 1)^2}\right]\right.$$

$$\left. + \sin(2n + 1)\theta \sin 2\theta \left[\frac{2 \cosh 2\beta + (2n + 1)\sinh 2\beta}{4 - (2n + 1)^2}\right] + \frac{\mu \cos(2n + 1)\theta}{2n + 1}\right\} \tag{9a}$$

$$y = \frac{b}{L}\sum_{n=0}^{\infty} P_n(\mu)e^{-(2n+1)\beta}\left\{- \sin(2n + 1)\theta \cos 2\theta \left[\frac{2 \sinh 2\beta + (2n + 1)\cosh 2\beta}{4 - (2n + 1)^2}\right]\right.$$

$$\left. + \cos(2n + 1)\theta \sin 2\theta \left[\frac{2 \cosh 2\beta + (2n + 1)\sinh 2\beta}{4 - (2n + 1)^2}\right] - \frac{\mu \sin(2n + 1)\theta}{2n + 1}\right\} \tag{9b}$$

On the surface, the series expansion involving the Legendre polynomials cannot be used, and the differential transformation equation of reference 3 (that is, eq. (6)) must be integrated as follows:

$$\frac{d\zeta}{d\tau} = F'\left[(2 \cos 2\tau - 2\mu)^{1/2}\right]_{\beta=0} \tag{10}$$

$$\frac{d\zeta}{d\theta} = F'(2 \cos 2\theta - 2\mu)^{1/2} \tag{11}$$

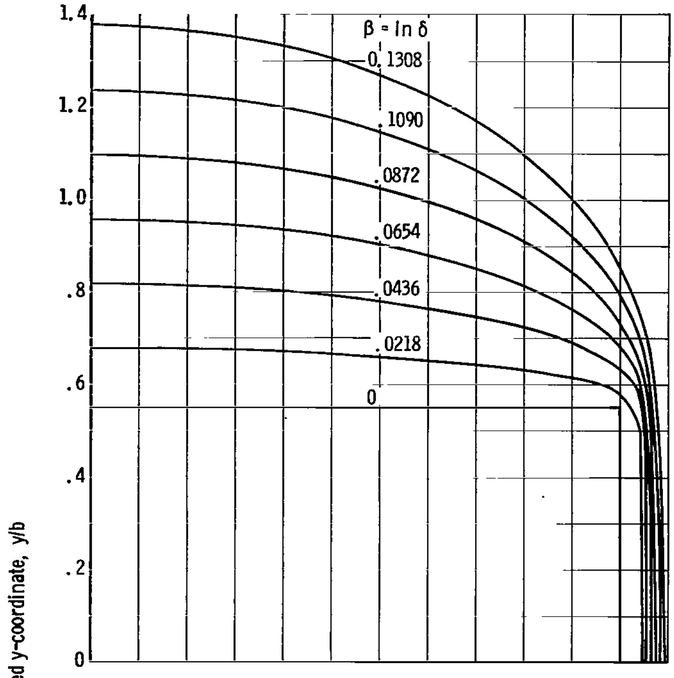
Integration of this differential equation using elliptic forms leads to the following:

For sides BC and AD:

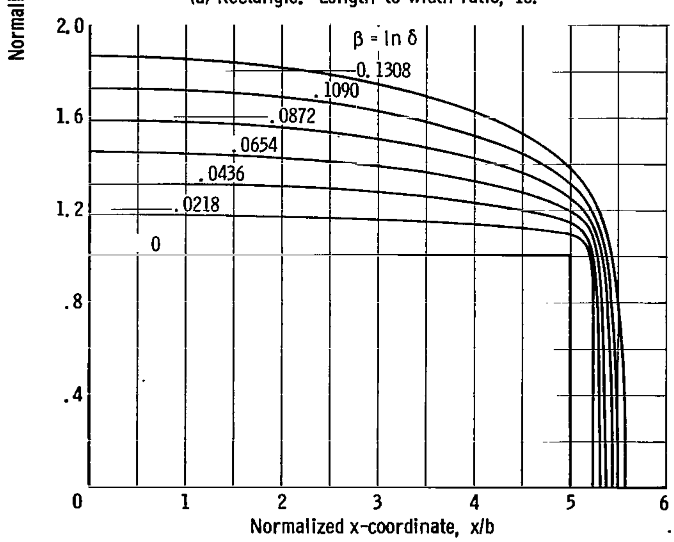
$$\zeta = 2F'\left[E(\theta, \sin \alpha) - \cos^2\alpha \; F(\theta, \sin \alpha)\right] \tag{12a}$$

where

$$-\alpha \leq \theta \leq \alpha \text{ and } \pi - \alpha \leq \theta \leq \pi + \alpha$$

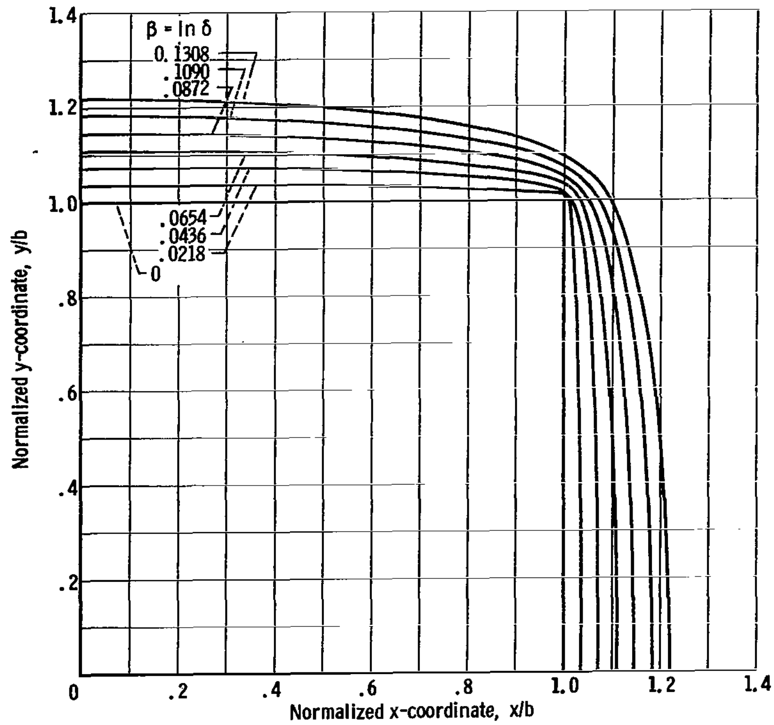


(a) Rectangle. Length-to-width ratio, 10.



(b) Rectangle. Length-to-width ratio, 5.

Figure 3. - Lines of constant magnetic potential around corner.



(c) Square.

Figure 3. - Concluded.

For sides AB and CD:

$$\zeta = -2iF'\left[E(\theta, \cos \alpha) - \sin^2\alpha \; F(\theta, \cos \alpha)\right] \tag{12b}$$

where

$$\alpha \leq \theta \leq \pi - \alpha \quad \text{and} \quad \pi + \alpha \leq \theta \leq - \alpha$$

In these two equations, the constant of integration has been set equal to zero so as to keep the rectangle centered at the origin of the ζ-space. $E(\theta, k)$ is the incomplete elliptic integral of the second kind, and $F(\theta, k)$ is the incomplete elliptic integral of the first kind. Both can be found in the appendix of reference 6 (note that $k = \sin \alpha$ or $\cos \alpha$). The fundamental net, which corresponds to the lines of constant magnetic potential and the lines of magnetic flux, can be plotted from equations (9a) and (9b). The lines of constant magnetic potential correspond to the family of constant β, and the lines of flux to the family of constant θ. The lines of constant magnetic potential for three rectangles (length-to-width ratios of 10:1, 5:1, and 1:1) are plotted in figure 3 around the corner of each rectangle for the same values of β.

Current Distribution

The use of equation (5) and the charge distribution of reference 3 shows that the normalized distribution of surface current per unit width in the region $-\alpha \leq \theta \leq \alpha$, which maps into the side BC (fig. 1), is given by

$$\frac{J_{z,I}}{\bar{J}} = \frac{\frac{2L}{\pi}\left(\frac{a}{b}+1\right)\sqrt{2}}{\sqrt{\sin^2\alpha - \sin^2\theta}} \tag{13}$$

where

$$\bar{J} = \frac{I}{4(a+b)}$$

In the region $\alpha \leq \theta \leq \pi - \alpha$ that represents side AB in figure 1, the current per unit width becomes

$$\frac{J_{z,II}}{\bar{J}} = \frac{\frac{2L}{\pi}\left(\frac{a}{b}+1\right)\sqrt{2}}{\sqrt{\sin^2\theta - \sin^2\alpha}} \tag{14}$$

The other two sides of the rectangle (AD, DC) have the same corresponding current distribution because of the symmetry. The current per unit width becomes infinite at the corners ($\theta = \pm\alpha$, $\pi\pm\alpha$) because the magnetic field goes to infinity at the corners. The total current, however, remains finite. The profile of this current density distribution across the conductor shows qualitatively the behavior of real currents flowing through superconductors and through normal conductors at high frequencies. The important point is that the highest concentration of current is around the corners.

This behavior of current per unit width at the corners is inherent in problems concerned with pure surface currents. In actual superconductors, where the current at the corners does not go to infinity, a deep penetration of the magnetic field into the sample occurs that causes the current to flow in a thicker layer along each corner.

Magnetic Field Distribution

The magnetic field can be obtained explicitly from the relation between the complex functions involved. A complex potential function $P(\theta, \beta)$ is sought in the t-space and then transformed into the ζ-space. When $P(\theta, \beta)$ is obtained for the ζ-space, the magnetic field is then obtained by the following transformation:

$$\vec{B} = -C_0 \hat{k} \times \vec{T} \tag{15}$$

where $\vec{T}$ is a vector in which the real and imaginary parts of $(dP/d\zeta)^*$ are the components in the x and y directions, respectively:

$$T_x = \text{real}\left[\left(\frac{dP}{d\zeta}\right)^*\right] \tag{16a}$$

$$T_y = \text{imag}\left[\left(\frac{dP}{d\zeta}\right)^*\right] \tag{16b}$$

The complex potential function is written as follows:

$$P(\theta, \beta) = U(\theta, \beta) + iV(\theta, \beta) \tag{17}$$

where in the t-space, $U(\theta, \beta)$ is the magnetic potential, and $V(\theta, \beta)$ represents the lines of magnetic flux. The complex solution of Laplace's equation in the t-space is

$$P(\theta, \beta) = \ln t = \ln \delta + i\theta \tag{18}$$

where $t = \delta\, e^{i\theta} = e^{i\tau} = e^{\beta + i\theta}$. The potential solution for the unit circle of the t-space is $U = \ln \delta$; $V = \theta$ represents the stream function. The magnetic field due to the normalized current of equations (13) and (14) may now be found from equations (6), (15), and (18):

$$\vec{B} = -G_o \hat{k} \times \overrightarrow{\left(\frac{dP}{dt}\frac{dt}{d\tau}\frac{d\tau}{d\zeta}\right)^*} = G_o \hat{k} \times \left[\frac{1}{iF'(2 \cos 2\tau - 2\mu)^{1/2}}\right]^* \tag{19}$$

where

$$G_o = \frac{\mu_o 2(a + b)}{\pi}$$

Substitution of $\tau = \theta - i\beta$ and $\mu = \cos 2\alpha$ into equation (19) leads to

$$\vec{B} = G_o \hat{k} \times \left\{ \frac{1}{\sqrt{2}\, iF} \frac{\left[(\cos 2\theta \cosh 2\beta - \cos 2\alpha) - i(\sin 2\theta \sinh 2\beta)\right]^{1/2}}{\left[(\cos 2\theta \cosh 2\beta - \cos 2\alpha)^2 + (\sin 2\theta \sinh 2\beta)^2\right]^{1/2}} \right\}^* \tag{20}$$

Letting

$$\left.\begin{aligned} M &= \cos 2\theta \cosh 2\beta - \cos 2\alpha \\ \text{and} \quad & \\ N &= \sin 2\theta \sinh 2\beta \end{aligned}\right\} \tag{21}$$

yields the following simplified expression for $\vec{B}$:

$$\vec{B} = G_o \hat{k} \times \left[\frac{1}{\sqrt{2}\, iF} \frac{(M - iN)^{1/2}}{(M^2 + N^2)^{1/2}} \right]^* \tag{22}$$

The ξ-space must be divided into two regions that depend on the value of M to determine the value of $(M - iN)^{1/2}$. In region I, let $M \geq 0$. Several algebraic manipulations then yield the following expressions for the magnetic field components:

$$B_{x,I} = \frac{\sqrt{2}\, G_o L}{b} \left[(\cos 2\theta \cosh 2\beta - \cos 2\alpha)^2 + (\sin 2\theta \sinh 2\beta)^2\right]^{-1/4}$$

$$\times \left[\sin\left(\frac{1}{2} \tan^{-1} \frac{\sin 2\theta \sinh 2\beta}{\cos 2\theta \cosh 2\beta - \cos 2\alpha}\right)\right] \tag{23a}$$

$$B_{y,I} = -\frac{\sqrt{2}\, G_o L}{b}\left[(\cos 2\theta \cosh 2\beta - \cos 2\alpha)^2 + (\sin 2\theta \sinh 2\beta)^2\right]^{-1/4}$$

$$\times \left[\cos\left(\frac{1}{2}\tan^{-1}\frac{\sin 2\theta \sinh 2\beta}{\cos 2\theta \cosh 2\beta - \cos 2\alpha}\right)\right] \tag{23b}$$

In region II let $M \leq 0$. The magnetic field components are obtained in region II in a similar way:

$$B_{x,II} = +\frac{G_o \sqrt{2}\, L}{b}\left[(\cos 2\theta \cosh 2\beta - \cos 2\alpha)^2 + (\sin 2\theta \sinh 2\beta)^2\right]^{-1/4}$$

$$\times \left[\cos\left(\frac{1}{2}\tan^{-1}\frac{\sin 2\theta \sinh 2\beta}{-\cos 2\theta \cosh 2\beta + \cos 2\alpha}\right)\right] \tag{24a}$$

$$B_{y,II} = -\frac{G_o \sqrt{2}\, L}{b}\left[(\cos 2\theta \cosh 2\beta - \cos 2\alpha)^2 + (\sin 2\theta \sinh 2\beta)^2\right]^{-1/4}$$

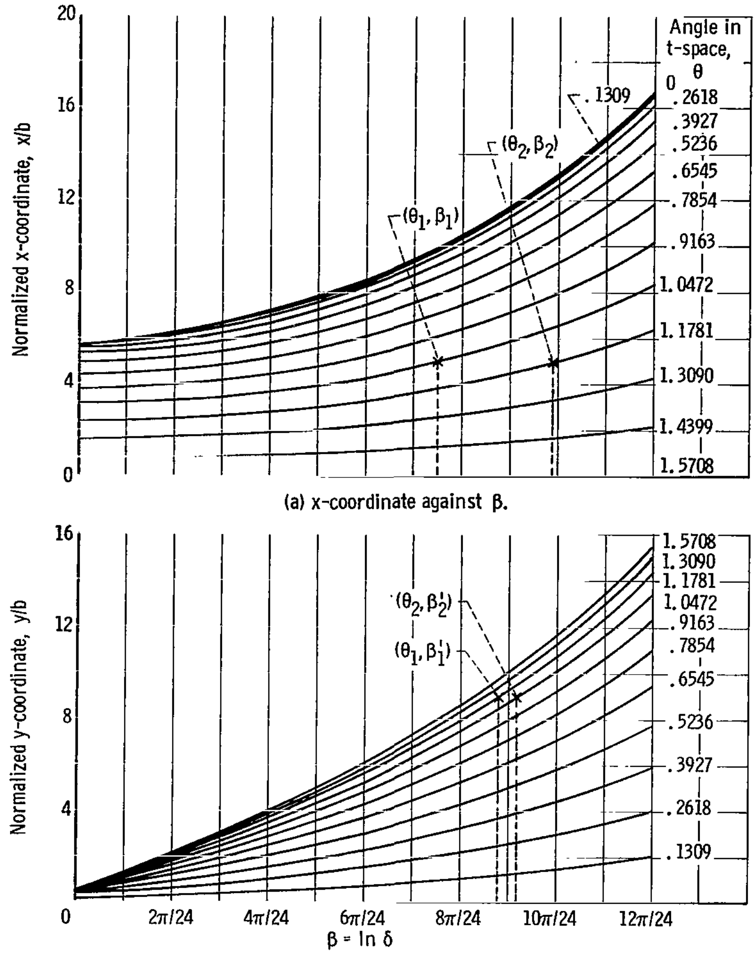
$$\times \left[\sin\left(\frac{1}{2}\tan^{-1}\frac{\sin 2\theta \sinh 2\beta}{-\cos 2\theta \cosh 2\beta + \cos 2\alpha}\right)\right] \tag{24b}$$

The absolute value of the magnetic field is

$$|B|^2 = \frac{2G_o^2 L^2}{b^2}\left[(\cos 2\theta \cosh 2\beta - \cos 2\alpha)^2 + (\sin 2\theta \sinh 2\beta)^2\right]^{-1/2}$$

The magnetic field is tangent to the equipotential lines at all points. Direct substitution into equations (23) and (24) gives for the surface of the conductor ($\beta = 0$) in region I, $B_{x,I} = 0$ and $B_{y,I} \propto \vec{J}$; for any $\beta \neq 0$ at $\theta = 0$ and at $\theta = \pi$, $B_{x,I} = 0$ and $B_{y,I} = |B|$. In region II for $\beta = 0$, $B_{x,II} \propto \vec{J}$ and $B_{y,II} = 0$; for any $\beta \neq 0$ at $\theta = \pi/2$ and at $\theta = 3\pi/2$, $B_{x,II} = |B|$ and $B_{y,II} = 0$. At the border between regions I and II, $(\cos 2\theta \cosh 2\beta - \cos 2\alpha = 0)$, $B_{x,I} = B_{x,II}$ and $B_{y,I} = B_{y,II}$. At infinity the denominator dominates the expression, and both B_x and B_y go to zero.

Due to the complexity of equations (9a) and (9b), the magnetic field $\vec{B}$ cannot be expressed in terms of x and y, the real variables of the original space. When knowledge of the magnetic field strength is sought for a given region in space, two ways of obtaining an answer are available. First, a table of corresponding values between x, y, and θ, β may be calculated from equations (9a) and (9b) for the particular width-to-length ratio in



(a) x-coordinate against β.

(b) y-coordinate against β.

Figure 4. - Coordinates x and y against β for rectangle of length-to-width ratio of 10. (θ_1 and θ_2 are shown for a point (5, 9).)

question, and $\vec{B}$ can be found from these values of θ and β from equations (23) and (24). The second method is more easily accomplished and is demonstrated as follows. Due to the symmetry of the rectangle, only the first quadrant $0 \leq \theta \leq \pi/2$ need be considered. Points outside the first quadrant can easily be rotated into the first quadrant by subtracting $n\pi/2$, with n depending on the location of the point. Therefore, plots of x and y against β for a family of constant θ's between zero and $\pi/2$ are obtained from equations (9a) and (9b) (fig. 4). For a given point, R(x, y), two values of θ (e.g., θ_1 and θ_2) are picked from the x and y graphs such that the corresponding β values to each θ_1 and θ_2 are nearly equal. This is shown in figure 4 for the point R(5, 9). The values obtained for θ_1 and θ_2 are 1.04 and 1.17, respectively. Once θ_1 and θ_2 are obtained, new plots of x and y against β can be calculated for a family of constant θ's between θ_1 and θ_2. New limits on θ are obtained, and the process is repeated until complex point $R'(\theta, \beta)$ gives a point $R'(x', y')$ within the required accuracy of the original point $R(x, y)$. Usually, the process need not be repeated many times, since, in practical applications, values of magnetic field are only sought for given areas, such as locations of wires or circuit components, rather than mathematical points.

CONCLUDING REMARKS

The magnetic field of a rectangular conductor, whose current flows on the surface, was obtained by the use of a complex transformation and the corresponding two-dimensional electrostatic potential solution. The results can be applied to any rectangular conductor whose cross-sectional dimensions are large compared with the magnetic field penetration distance so that the induced current can be assumed to flow on the surface. Such systems are waveguides, good conductors with alternating currents, and superconductors whose cross-sectional dimensions are much larger than the London penetration depth.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, May 24, 1966,
129-02-05-02-22.

APPENDIX - SYMBOLS

$\vec{A}$	magnetic vector potential
A_z	component of $\vec{A}$ along z-coordinate axis
a	width of rectangular conductor
$\vec{B}$	magnetic field induction
$B_{x,I}$	component of $\vec{B}$ along x-coordinate axis in region I
$B_{x,II}$	component of $\vec{B}$ along x-coordinate axis in region II
$B_{y,I}$	component of $\vec{B}$ along y-coordinate axis of region I
$B_{y,II}$	component of $\vec{B}$ along y-coordinate axis in region II
b	thickness of rectangular conductor
C_0	1 Wb/(V)(m)
D	constant of integration
dv'	volume element of conductor
$\vec{E}$	electrostatic field
E(k)	complete elliptic integral of 2nd kind
E'	E(k')
F'	$-\frac{1}{4}C \csc \alpha = \frac{ib}{2L}$
G_o	$\frac{\mu_o 2(a + b)}{\pi}$
I	total current passing through conductor
$\vec{J}$	surface current density in the conductor
J_z	component of surface current density along z-coordinate axis
K(k)	complete elliptic integral of first kind
K'	K(k')
k	modulus of Jacobian elliptic functions and integrals, $\sin \alpha$
k'	complementary modulus, $\sqrt{1 - k^2}$
$\hat{k}$	unit vector along z-coordinate axis
L	$E(k) - k'^2 K$
L'	$E(k') - k^2 K'$
M	$\cos 2\theta \cosh 2\beta - \cos 2\alpha$
N	$\sin 2\theta \sinh 2\beta$
$\hat{n}$	unit vector perpendicular to surface
$P(\theta, \beta)$	complex potential function
$P_n(\mu)$	Legendre polynomial
R(x, y)	sample point in calculation of transformation
$\vec{r}$	radial coordinate of field point, $r^2 = x^2 + y^2 + z^2$
$\vec{r}'$	radial coordinate of source point, $r'^2 = x'^2 + y'^2 + z'^2$
$\vec{T}$	vector whose components are real and imaginary parts of $(dP/d\zeta)^*$

t	complex space where conductor is circular, $t = e^{i\tau} = \delta e^{i\theta}$
x, y, z	coordinate axes of ζ-space
α	angle in t-space corresponding to corners of rectangle
β	$\ln \delta$
δ	radial vector in t-space
ϵ	permitivity
ϵ_o	permitivity of free space
ζ	complex space where conductor is rectangular, $\zeta = x + iy$
θ	angle in t-space
κ_1, κ_2	dielectric constants
μ	$\cos 2\alpha$
μ_o	magnetic permeability of free space
μ_1, μ_2	magnetic permeability for regions 1 and 2
$\rho(x', y')$	electrostatic charge distribution
σ	surface-charge density
τ	$\ln t = \theta - i\beta$
φ	electrostatic potential

Subscripts:

1, 2	regions on either side of boundary between two materials
I	region I, $\cos 2\theta \cosh 2\beta - \cos 2\alpha \geq 0$
II	region II, $\cos 2\theta \cosh 2\beta - \cos 2\alpha \leq 0$

Superscript:

*	denotes complex conjugate functions

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